EGOROV'S THEOREM FOR TRANSVERSALLY ELLIPTIC OPERATORS ON FOLIATED MANIFOLDS AND NONCOMMUTATIVE GEODESIC FLOW

YURI A. KORDYUKOV

Abstract. The main result of the paper is Egorov's theorem for transversally elliptic operators on compact foliated manifolds. This theorem is applied to describe the noncommutative geodesic flow in noncommutative geometry of Riemannian foliations.

Introduction

Egorov's theorem [8] is one of the fundamental results in microlocal analysis that relates the quantum evolution of pseudodifferential operators with the classical dynamics of principal symbols.

Let P be a positive, self-adjoint, elliptic, first order pseudodifferential operator on a compact manifold M with the positive principal symbol $p \in S^1(T^*M\setminus 0)$. Let f_t be the bicharacteristic flow of the operator P, that is, the Hamiltonian flow of p on T^*M. For instance, one can consider $P = \sqrt{\Delta_M}$, where Δ_M is the Laplace operator of a Riemannian metric g_M on M. Then the bicharacteristic flow of the operator P is the geodesic flow of the metric g_M.

Egorov's theorem states that, for any pseudodifferential operator A of order 0 with the principal symbol $a \in S^0(T^*M \setminus 0)$, the operator $A(t) = e^{itP}Ae^{-itP}$ is a pseudodifferential operator of order 0. The principal symbol $a_t \in S^0(T^*M \setminus 0)$ of this operator is given by the formula

$$a_t(x,\xi) = a(f_t(x,\xi)), \quad (x,\xi) \in T^*M \setminus 0.$$

The main result of this paper is a version of Egorov's theorem for transversally elliptic operators on compact foliated manifolds. This theorem is applied to describe the noncommutative geodesic flow in noncommutative geometry of Riemannian foliations.

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1. Preliminaries and main results

1.1. **Transverse pseudodifferential calculus.** Throughout in the paper, $(M,\mathcal{F})$ is a compact foliated manifold, E is a Hermitian vector bundle on M, $\dim M = n, \dim\mathcal{F} = p, p+q = n$.

We will consider pseudodifferential operators, acting on half-densities. For any vector bundle V on M, denote by $|V|^{1/2}$ the associated half-density vector bundle. Let $C^\infty(M,E)$ denote the space of smooth sections of the vector bundle $E\otimes|TM|^{1/2}$, $L^2(M,E)$ the Hilbert space of square integrable sections of $E\otimes|TM|^{1/2}$, $\mathcal{D}'(M,E)$ the space of distributional sections of $E\otimes|TM|^{1/2}$, $\mathcal{D}'(M,E) = C^\infty(M,E)'$, and $H^s(M,E)$ the Sobolev space of order s of sections of $E\otimes|TM|^{1/2}$. Finally, let $\Psi^m(M,E)$ denote the standard classes of pseudodifferential operators, acting in $C^\infty(M,E)$.

We will use the classes $\Psi^{m,-\infty}(M,\mathcal{F},E)$ of transversal pseudodifferential operators. Let us briefly recall its definition, referring the reader to [14] for more details.

We will consider foliated coordinate charts $\varkappa : U\subset M \xrightarrow{\sim} I^n$ on M with coordinates $(x,y)\in I^p\times I^q$ (I is the open interval $(0,1)$) such that the restriction of $\mathcal{F}$ to U is given by the sets $y = \text{const}$. We will always assume that foliated charts are regular. Recall that a foliated coordinate chart $\varkappa : U\subset M \xrightarrow{\sim} I^n$ is called regular, if it admits an extension to a foliated coordinate chart $\bar\varkappa : V\subset M \xrightarrow{\sim} (-2,2)^n$ with $\bar U\subset V$.

A map $f : U\subset M\to\mathbb{R}^q$ is called a distinguished map, if f locally has the form $pr_{nq}\circ\varkappa$, where $\varkappa : V\subset U \xrightarrow{\sim} I^n$ is a foliated chart and $pr_{nq} : \mathbb{R}^n = \mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^q$ is the natural projection. Let D_x denote the set of germs of distinguished maps from M to $\mathbb{R}^q$ at a point $x\in M$. For any leafwise continuous curve γ from x to y, let $h_\gamma : D_x\to D_y$ be the holonomy map associated with γ. This is the generalization of Poincaré's first return map from flows to foliations.

Let $\varkappa : U\to I^p\times I^q, \varkappa' : U'\to I^p\times I^q$, be two foliated charts, $\pi = pr_{nq}\circ\varkappa : U\to\mathbb{R}^q$, $\pi' = pr_{nq}\circ\varkappa' : U'\to\mathbb{R}^q$ the corresponding distinguished maps. The foliation charts $\varkappa$, $\varkappa'$ are called compatible, if, for any $m\in U$ and $m'\in U'$ such that $m = \varkappa^{-1}(x,y)$, $m' = \varkappa'^{-1}(x',y)$ with the same y, there is

a leafwise path γ from m to m' such that the corresponding holonomy map h_γ takes the germ π_m of the map π at m to the germ $\pi'_{m'}$ of the map π' at m'.

Let $\varkappa : U \subset M \to I^p \times I^q, \varkappa' : U' \subset M \to I^p \times I^q$, be two compatible foliated charts on M equipped with trivializations of the vector bundle E over them. Consider an operator $A : C^\infty_c(U, E|_U) \to C^\infty_c(U', E|_{U'})$ given in the local coordinates by the formula

$$Au(x,y) = (2\pi)^{-q} \int e^{i(y-y')\eta} k(x,x',y,\eta) u(x',y')\, dx'\, dy'\, d\eta, \tag{1}$$

where $k \in S^m(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$, $u \in C^\infty_c(I^n, \mathbb{C}^r), x \in I^p, y \in I^q$ with the Schwartz kernel, compactly supported in $U \times U'$ (here $r = \operatorname{rank} E$).

Recall that a function $k \in C^\infty(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$ belongs to the class $S^m(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$, if, for any multiindices α and β, there exists a constant $C_{\alpha\beta} > 0$ such that

$$|\partial^\alpha_\eta \partial^\beta_{(x,x',y)} k(x,x',y,\eta)| \leq C_{\alpha\beta}(1+|\eta|)^{m-|\alpha|},$$
$$(x,x',y) \in I^p \times I^p \times I^q, \quad \eta \in \mathbb{R}^q.$$

We will consider only classical symbols k, which can be represented as an asymptotic sum $k(x,x',y,\eta) \sim \sum_{j=0}^\infty \theta(\eta) k_{z-j}(x,x',y,\eta)$, where $k_{z-j} \in C^\infty(I^p \times I^p \times I^q \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(\mathbb{C}^r))$ is homogeneous in η of degree $z-j$, and θ is a smooth function on $\mathbb{R}^q$ such that $\theta(\eta) = 0$ for $|\eta| \leq 1$, $\theta(\eta) = 1$ for $|\eta| \geq 2$.

The operator A extends to an operator in $C^\infty(M,E)$ in a trivial way. The resulting operator is called an elementary operator of class $\Psi^{m,-\infty}(M,\mathcal{F},E)$.

The class $\Psi^{m,-\infty}(M,\mathcal{F},E)$ consists of all operators A in $C^\infty(M,E)$, which can be represented in the form $A = \sum_{i=1}^k A_i + K$, where A_i are elementary operators of class $\Psi^{m,-\infty}(M,\mathcal{F},E)$, corresponding to some pairs $\varkappa_i, \varkappa'_i$ of compatible foliated charts, $K \in \Psi^{-\infty}(M,E)$. Put $\Psi^{*,-\infty}(M,\mathcal{F},E) = \bigcup_m \Psi^{m,-\infty}(M,\mathcal{F},E)$.

Let G be the holonomy groupoid of $\mathcal{F}$. We will briefly recall its definition. Let $\sim_h$ be the equivalence relation on the set of continuous leafwise paths $\gamma : [0,1] \to M$, setting $\gamma_1 \sim_h \gamma_2$ if γ_1 and γ_2 have the same initial and final points and the same holonomy maps. The holonomy groupoid G is the set of $\sim_h$ equivalence classes of continuous leafwise paths. G is equipped with the source and the range maps $s, r : G \to M$ defined by $s(\gamma) = \gamma(0)$ and $r(\gamma) = \gamma(1)$. We will identify a point $x \in M$ with the element of G given by the corresponding constant path: $\gamma(t) = x, t \in [0,1]$. Recall also that, for any $x \in M$, the set $G^x = \{\gamma \in G : r(\gamma) = x\}$ is the covering of the leaf through the point x associated with the holonomy group G^x_x of this leaf, $G^x_x = \{\gamma \in G : s(\gamma) = x, r(\gamma) = x\}$.

Any pair of compatible foliated charts $\varkappa : U \to I^p \times I^q, \varkappa' : U' \to I^p \times I^q$ defines a foliated chart $V \to I^p \times I^p \times I^q$ on G as follows. The coordinate patch V consists of all $\gamma \in G$ from $m = \varkappa^{-1}(x,y) \in U$ to $m' = \varkappa'^{-1}(x',y) \in$

U' such that the corresponding holonomy map h_γ takes the germ π_m of the distinguished map $\pi = pr_{nq} \circ \varkappa$ at m to the germ $\pi'_{m'}$ of the distinguished map $\pi' = pr_{nq} \circ \varkappa'$ at m', and the coordinate map takes such a γ to $(x, x', y) \in I^p \times I^p \times I^q$.

Denote by $N^*\mathcal{F}$ the conormal bundle to $\mathcal{F}$. For any $\gamma \in G, s(\gamma) = x, r(\gamma) = y$, the codifferential of the corresponding holonomy map defines a linear map $dh^*_\gamma : N^*_y\mathcal{F} \to N^*_x\mathcal{F}$. Let $\mathcal{F}_N$ be the linearized foliation in $\tilde{N}^*\mathcal{F} = N^*\mathcal{F} \setminus 0$ (cf., for instance, [20]). The leaf of the foliation $\mathcal{F}_N$ through $\nu \in \tilde{N}^*\mathcal{F}$ is the set of all points $dh^*_\gamma(\nu) \in \tilde{N}^*\mathcal{F}$, where $\gamma \in G, r(\gamma) = \pi(\nu)$ (here $\pi : T^*M \to M$ is the bundle map). The leaves of the foliation $\mathcal{F}_N$ have trivial holonomy. Therefore, the holonomy groupoid $G_{\mathcal{F}_N}$ of $\mathcal{F}_N$ consists of all pairs $(\gamma, \nu) \in G \times \tilde{N}^*\mathcal{F}$ such that $r(\gamma) = \pi(\nu)$ with the source map $s_N : G_{\mathcal{F}_N} \to \tilde{N}^*\mathcal{F}, s_N(\gamma, \nu) = dh^*_\gamma(\nu)$ and the range map $r_N : G_{\mathcal{F}_N} \to \tilde{N}^*\mathcal{F}, r_N(\gamma, \nu) = \nu$. We have a map $\pi_G : G_{\mathcal{F}_N} \to G$ given by $\pi_G(\gamma, \nu) = \gamma$.

Denote by π^*E the lift of the vector bundle E to $\tilde{N}^*\mathcal{F}$ via the bundle map $\pi : \tilde{N}^*\mathcal{F} \to M$ and by $\mathcal{L}(\pi^*E)$ the vector bundle on $G_{\mathcal{F}_N}$, whose fiber at a point $(\gamma, \nu) \in G_{\mathcal{F}_N}$ is the space $\mathcal{L}((\pi^*E)_{s_N(\gamma,\nu)}, (\pi^*E)_{r_N(\gamma,\nu)})$ of linear maps from $(\pi^*E)_{s_N(\gamma,\nu)}$ to $(\pi^*E)_{r_N(\gamma,\nu)}$. There is a natural foliation $\mathcal{G}_N$ on $G_{\mathcal{F}_N}$. The leaf of $\mathcal{G}_N$ through a point $(\gamma, \nu) \in G_{\mathcal{F}_N}$ is the set of all $(\gamma', \nu') \in G_{\mathcal{F}_N}$ such that ν and ν' lie in the same leaf in $\mathcal{F}_N$. Let $|T\mathcal{G}_N|^{1/2}$ be the line bundle of leafwise half-densities on $G_{\mathcal{F}_N}$ with respect to the foliation $\mathcal{G}_N$. It is easy to see that

$$|T\mathcal{G}_N|^{1/2} = r^*_N(|T\mathcal{F}_N|^{1/2}) \otimes s^*_N(|T\mathcal{F}_N|^{1/2}),$$

where $s^*_N(|T\mathcal{F}_N|^{1/2})$ and $r^*_N(|T\mathcal{F}_N|^{1/2})$ denote the lifts of the line bundle $|T\mathcal{F}_N|^{1/2}$ of leafwise half-densities on $N^*\mathcal{F}$ via the source and the range mappings s_N and r_N respectively.

A section $k \in C^\infty(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ is said to be properly supported, if the restriction of the map $r : G_{\mathcal{F}_N} \to \tilde{N}^*\mathcal{F}$ to $\operatorname{supp} k$ is a proper map. Consider the space $C^\infty_{prop}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ of smooth, properly supported sections of $\mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2}$. One can introduce the structure of involutive algebra on $C^\infty_{prop}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ by the standard formulas (cf. (8)). Let $S^m(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ be the space of all $s \in C^\infty_{prop}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ homogeneous of degree m with respect to the action of $\mathbb{R}$ given by the multiplication in the fibers of the vector bundle $\pi_G : G_{\mathcal{F}_N} \to G$. By [14], there is the half-density principal symbol mapping

$$\sigma : \Psi^{m,-\infty}(M, \mathcal{F}, E) \to S^m(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2}), \tag{2}$$

which satisfies

$$\sigma_{m_1+m_2}(AB) = \sigma_{m_1}(A)\sigma_{m_2}(B), \quad \sigma_{m_1}(A^*) = \sigma_{m_1}(A)^*$$

for any $A \in \Psi^{m_1,-\infty}(M, \mathcal{F}, E)$ and $B \in \Psi^{m_2,-\infty}(M, \mathcal{F}, E)$.

Example 1.1. Consider a foliated coordinate chart $\varkappa : U \subset M \xrightarrow{\sim} I^n$ on M with coordinates $(x,y) \in I^p \times I^q$. One has the corresponding coordinate chart in T^*M with coordinates given by $(x,y,\xi,\eta) \in I^p \times I^q \times \mathbb{R}^p \times \mathbb{R}^q$. In these coordinates, the restriction of the conormal bundle $N^*\mathcal{F}$ to U is given by the equation $\xi = 0$. So we have a coordinate chart $\varkappa_n : U_1 \subset N^*\mathcal{F} \xrightarrow{\sim} I^p \times I^q \times \mathbb{R}^q$ on $N^*\mathcal{F}$ with the coordinates $(x,y,\eta) \in I^p \times I^q \times \mathbb{R}^q$. The coordinate chart $\varkappa_n$ is a foliated coordinate chart for the linearized foliation $\mathcal{F}_N$, and the restriction of $\mathcal{F}_N$ to U_1 is given by the level sets $y = \text{const}, \eta = \text{const}$.

Now let $\varkappa : U \subset M \to I^p \times I^q, \varkappa' : U' \subset M \to I^p \times I^q$, be two compatible foliated charts on M. Then the corresponding foliated charts $\varkappa_n : U_1 \subset N^*\mathcal{F} \to I^p \times I^q \times \mathbb{R}^q, \varkappa'_n : U'_1 \subset N^*\mathcal{F} \to I^p \times I^q \times \mathbb{R}^q$, are compatible with respect to the foliation $\mathcal{F}_N$. So they define a foliated chart V on the foliated manifold $(G_{\mathcal{F}_N}, \mathcal{G}_N)$ with the coordinates $(x,x',y,\eta) \in I^p \times I^p \times I^q \times \mathbb{R}^q$, and the restriction of $\mathcal{G}_N$ to V is given by the level sets $y = \text{const}, \eta = \text{const}$. The principal symbol $\sigma_m(A)$ of an operator A given by the formula (1) is the half-density $k_m(x,x',y,\eta)\,|dx|^{1/2}\,|dx'|^{1/2}$, where k_m is the top degree homogeneous component of k. It can be checked that this half-density is globally defined as an element of the space $S^m(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$.

1.2. **Transverse bicharacteristic flow.** For any operator $P \in \Psi^m(M,E)$, let σ_P denote the transversal principal symbol of P, which is the restriction of its principal symbol to $\tilde{N}^*\mathcal{F}$. We say that P is transversally elliptic, if $\sigma_P(\nu)$ is invertible for any $\nu \in \tilde{N}^*\mathcal{F}$.

Consider a transversally elliptic operator $A \in \Psi^2(M,E)$ which has the scalar principal symbol and the holonomy invariant transverse principal symbol. Here the holonomy invariance of the transversal principal symbol $\sigma_A \in C^\infty(\tilde{N}^*\mathcal{F})$ means that it is constant along the leaves of the foliation $\mathcal{F}_N$:

$$\sigma_A(dh^*_\gamma(\nu)) = \sigma_A(\nu), \quad \gamma \in G, \quad \nu \in N^*_{r(\gamma)}\mathcal{F}.$$

Let $a_2 \in S^2(\tilde{T}^*M)$ be the principal symbol of A. (Here $\tilde{T}^*M = T^*M \setminus 0$). Take any scalar elliptic symbol $\tilde{p} \in S^1(\tilde{T}^*M)$, which is equal to $\sqrt{a_2}$ in some conic neighborhood of $\tilde{N}^*\mathcal{F}$. Denote by $X_{\tilde{p}}$ the Hamiltonian vector field of $\tilde{p}$ on T^*M. Since $N^*\mathcal{F}$ is a coisotropic submanifold in T^*M and $T\mathcal{F}_N$ is the symplectic orthogonal complement of $T(N^*\mathcal{F})$, one can show that $X_{\tilde{p}}$ is tangent to $\tilde{N}^*\mathcal{F}$, and its restriction to $\tilde{N}^*\mathcal{F}$ (denoted also by $X_{\tilde{p}}$) is an infinitesimal transformation of the foliation $\mathcal{F}_N$, i.e. for any vector field X on $\tilde{N}^*\mathcal{F}$, tangent to $\mathcal{F}_N$, the commutator $[X_{\tilde{p}}, X]$ is tangent to $\mathcal{F}_N$. It follows that the Hamiltonian flow $\tilde{f}_t$ of $\tilde{p}$ preserves $\tilde{N}^*\mathcal{F}$, and its restriction to $N^*\mathcal{F}$ (denoted by f_t) preserves the foliation $\mathcal{F}_N$, that is, takes any leaf of $\mathcal{F}_N$ to a leaf.

Let $\tau = TN^*\mathcal{F}/T\mathcal{F}_N$ be the normal space to the foliation $\mathcal{F}_N$ and $\pi_{tr} : TN^*\mathcal{F} \to \tau$ the natural projection. For any $(\gamma,\nu) \in G_{\mathcal{F}_N}$, let $dH_{(\gamma,\nu)} : \tau_{dh^*_\gamma(\nu)} \to \tau_\nu$ be the corresponding linear holonomy map. The differential of

the map $(s_N, r_N) : G_{\mathcal{F}_N} \to N^*\mathcal{F} \times N^*\mathcal{F}$ at a point $(\gamma,\nu) \in G_{\mathcal{F}_N}$ defines an inclusion of $T_{(\gamma,\nu)}G_{\mathcal{F}_N}$ into $T_{dh^*_\gamma(\nu)}N^*\mathcal{F} \times T_\nu N^*\mathcal{F}$, and its image consists of all $(X,Y) \in T_{dh^*_\gamma(\nu)}N^*\mathcal{F} \times T_\nu N^*\mathcal{F}$ such that

$$\pi_{tr}(Y) = dH_{(\gamma,\nu)}(\pi_{tr}(X)). \tag{3}$$

Since $X_{\tilde p}$ is an infinitesimal transformation of the foliation $\mathcal{F}_N$, one can see that, for any $(\gamma,\nu) \in G_{\mathcal{F}_N}$, the pair $(X_{\tilde p}(dh^*_\gamma(\nu)), X_{\tilde p}(\nu)) \in T_{dh^*_\gamma(\nu)}N^*\mathcal{F} \times T_\nu N^*\mathcal{F}$ satisfies (3). Therefore, there exists a unique vector field $\mathcal{H}_p$ on $G_{\mathcal{F}_N}$ such that $ds_N(\mathcal{H}_p) = X_{\tilde p}$ and $dr_N(\mathcal{H}_p) = X_{\tilde p}$. Let F_t be the flow on $G_{\mathcal{F}_N}$ determined by the vector field $\mathcal{H}_p$. It is easy to see that $s_N \circ F_t = f_t \circ s_N$, $r_N \circ F_t = f_t \circ r_N$ and the flow F_t preserves the foliation $\mathcal{G}_N$.

Definition 1.2. Let $P = \sqrt{A}$ be an (unbounded) linear operator in $L^2(M,E)$, where $A \in \Psi^2(M,E)$ is an essentially self-adjoint, transversally elliptic operator, which has the scalar principal symbol and the holonomy invariant transverse principal symbol. **The transversal bicharacteristic flow** of P is the one-parameter group F^*_t of automorphisms of the involutive algebra $C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ induced by the flow F_t on $G_{\mathcal{F}_N}$.

Remark 1.3. It is easy to see that the definition of transversal bicharacteristic flow is independent of a choice of the elliptic extension $\tilde p$.

Example 1.4. Consider a foliated coordinate chart $\varkappa : U \subset M \xrightarrow{\sim} I^n$ on M with coordinates $(x,y) \in I^p \times I^q$. Let $\tilde p$ be a positive, smooth homogeneous of degree 1 function on $I^n \times (\mathbb{R}^n \setminus \{0\})$ (a scalar elliptic principal symbol) such that the corresponding transversal principal symbol σ_P is holonomy invariant. This means

$$\tilde p(x,y,0,\eta) = p(y,\eta), \quad x \in I^p, \quad y \in I^q, \quad \eta \in \mathbb{R}^q$$

with some function p. The Hamiltonian vector field $X_{\tilde p}$ on $I^n \times \mathbb{R}^n$ is given by

$$X_{\tilde p} = \frac{\partial \tilde p}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial \tilde p}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \tilde p}{\partial \eta}\frac{\partial}{\partial y} - \frac{\partial \tilde p}{\partial y}\frac{\partial}{\partial \eta},$$

and its restriction to $N^*\mathcal{F}\,|_U \cong I^p \times I^q \times \mathbb{R}^q$ is given by

$$X_{\tilde p}(x,y,\eta) = \frac{\partial \tilde p}{\partial \xi}(x,y,0,\eta)\frac{\partial}{\partial x} + \frac{\partial p}{\partial \eta}(y,\eta)\frac{\partial}{\partial y} - \frac{\partial p}{\partial y}(y,\eta)\frac{\partial}{\partial \eta}, \qquad (x,y,\eta) \in I^p \times I^q \times \mathbb{R}^q.$$

The fact that $X_{\tilde p}$ is an infinitesimal transformation of the foliation $\mathcal{F}_N$ means that its transverse part

$$\frac{\partial p}{\partial \eta}(y,\eta)\frac{\partial}{\partial y} - \frac{\partial p}{\partial y}(y,\eta)\frac{\partial}{\partial \eta}$$

is independent of x. The corresponding vector field $\mathcal{H}_p$ on $G_{\mathcal{F}_N}$ is given by

$$\mathcal{H}_p(x,x',y,\eta) = \frac{\partial\tilde{p}}{\partial\xi}(x,y,0,\eta)\frac{\partial}{\partial x} + \frac{\partial\tilde{p}}{\partial\xi}(x',y,0,\eta)\frac{\partial}{\partial x'} + \frac{\partial p}{\partial\eta}(y,\eta)\frac{\partial}{\partial y} - \frac{\partial p}{\partial y}(y,\eta)\frac{\partial}{\partial\eta}, \quad (x,x',y,\eta)\in I^p\times I^p\times I^q\times\mathbb{R}^q.$$

Finally, the transversal bicharacteristic flow is given by the action of the flow F_t determined by the vector field $\mathcal{H}_p$ on the space of half-densities of the form $k_m(x,x',y,\eta)\,|dx|^{1/2}\,|dx'|^{1/2}$.

Remark 1.5. The construction of the transversal bicharacteristic flow provides an example of what can be called noncommutative symplectic (or, maybe, better, Poisson) reduction. Here symplectic reduction means the following procedure [16, Chapter III, Section 14] (see also [17, 18]).

Let (X,ω) be a symplectic manifold, and Y a submanifold of X such that the 2-form ω_Y induced by ω on Y is of constant rank. Let $\mathcal{F}_Y$ be the characteristic foliation of Y relative to ω_Y. If the foliation $\mathcal{F}_Y$ is simple, that is, it is given by the fibers of a surjective submersion p of Y to a smooth manifold B, then B has a unique symplectic form ω_B such that $p^*\omega_B = \omega_Y$. The symplectic manifold (B,ω_B) is said to be the reduced symplectic manifold associated with Y. In a particular case when the submanifold Y is the preimage of a point under the momentum map associated with the Hamiltonian action of a Lie group, the symplectic reduction associated with Y is the Mardsen-Weinstein symplectic reduction [19].

Moreover (see, for instance, [16, Chapter III, Theorem 14.6]), if Y is invariant under the Hamiltonian flow of a Hamiltonian $H\in C^\infty(X)$ (this is equivalent to the fact that $(dH)\,|_Y$ is constant along the leaves of the characteristic foliation $\mathcal{F}_Y$), there exists a unique function $\hat{H}\in C^\infty(B)$, called the reduced Hamiltonian, such that $H\,|_Y = \hat{H}\circ p$. Furthermore, the map p projects the restriction of the Hamiltonian flow of H to Y to the reduced Hamiltonian flow on B defined by the reduced Hamiltonian $\hat{H}$.

Now let $(M,\mathcal{F})$ be a smooth foliated manifold. Consider the symplectic reduction associated with the coisotropic submanifold $Y = N^*\mathcal{F}$ in the symplectic manifold $X = T^*M$. The corresponding characteristic foliation $\mathcal{F}_Y$ is the linearized foliation $\mathcal{F}_N$. In general, the leaf space $N^*\mathcal{F}/\mathcal{F}_N$ is not a smooth manifold. Following ideas of the noncommutative geometry in the sense of A. Connes, one can treat the algebra $C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ as a noncommutative analogue of an algebra of smooth functions on $N^*\mathcal{F}/\mathcal{F}_N$. The symplectic reduction procedure is applied to the Hamiltonian flow $\tilde{f}_t$ of a function $\tilde{p}$ satisfying the assumptions given in the beginning of this section, yielding the transversal bicharacteristic flow F_t^* as the corresponding reduced Hamiltonian flow on $N^*\mathcal{F}/\mathcal{F}_N$. Following the ideas of [2, 28], one can interpret the algebra $C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ as a noncommutative Poisson manifold and the flow F_t^* as a noncommutative Hamiltonian flow.

Example 1.6. Let $(M,\mathcal{F})$ be a compact Riemannian foliated manifold equipped with a bundle-like metric g_M. Let $F=T\mathcal{F}$ be the tangent bundle to $\mathcal{F}$, H the orthogonal complement to F, and g_H the restriction of g_M to H. By definition, a Riemannian metric g_M on M is called bundle-like, if it satisfies one of the following equivalent conditions (see, for instance, [20, 23]):

1. For any continuous leafwise path γ from x to y, the corresponding linear holonomy map $dh_\gamma : T_xM/T_x\mathcal{F} \to T_yM/T_y\mathcal{F}$ is an isometry with respect to the Riemannian structures on $T_xM/T_x\mathcal{F}$ and $T_yM/T_y\mathcal{F}$ induced by the metric g_M;

2. If g_H is written as $g_H=\sum_{\alpha\beta} g_{\alpha\beta}(x,y)\theta^\alpha\theta^\beta$ in some foliated chart with coordinates $(x,y)\in I^p\times I^q$, where $\theta^\alpha\in H^*$ is the (unique) lift of dy^α under the projection $I^p\times I^q\to I^q$, then $g_{\alpha\beta}$ is independent of x, $g_{\alpha\beta}(x,y)=g_{\alpha\beta}(y)$.

The decomposition $F\oplus H=TM$ induces a bigrading on $\bigwedge T^*M$:

$$\bigwedge^k T^*M=\bigoplus_{i=0}^k \bigwedge^{i,k-i} T^*M,$$

where $\bigwedge^{i,j} T^*M=\bigwedge^i F^*\otimes\bigwedge^j H^*$. In this bigrading, the de Rham differential d can be written as

$$d=d_F+d_H+\theta,$$

where d_F and d_H are first order differential operators (the tangential de Rham differential and the transversal de Rham differential accordingly), and θ is a zero order differential operator.

The transverse signature operator is a first order differential operator in $C^\infty(M,\bigwedge H^*)$ given by

$$D_H=d_H+d_H^*,$$

and the transversal Laplacian is a second order transversally elliptic differential operator in $C^\infty(M,\bigwedge H^*)$ given by

$$\Delta_H=D_H^2.$$

The principal symbol $\sigma(\Delta_H)$ of Δ_H is given by

$$\sigma(\Delta_H)(x,\xi)=g_H(\xi,\xi)I_x,\quad (x,\xi)\in\tilde{T}^*M,$$

and holonomy invariance of the transversal principal symbol is equivalent to the assumption on the metric g_M to be bundle-like.

Take any function $p_2\in C^\infty(T^*M)$, which coincides with $\sqrt{\sigma(\Delta_H)}$ in some conical neighborhood of $N^*\mathcal{F}$. The restriction of the Hamiltonian flow of p_2 to $N^*\mathcal{F}$ coincides with the restriction G_t of the geodesic flow g_t of the Riemannian metric g_M to $N^*\mathcal{F}$, which is the transversal bicharacteristic flow of the operator $\langle D_H\rangle=\sqrt{\Delta_H+I}$.

Finally, if $\mathcal{F}$ is given by the fibers of a Riemannian submersion $f:M\to B$, then there is a natural isomorphism $N^*_m\mathcal{F}\to T^*_{f(m)}B$, and, under this

isomorphism, the transversal geodesic flow G_t on $N^*\mathcal{F}$ corresponds to the geodesic flow T^*B (see, for instance, [21, 23]).

1.3. **Egorov's theorem.** Let $D \in \Psi^1(M, E)$ be a formally self-adjoint, transversally elliptic operator such that D^2 has the scalar principal symbol and the holonomy invariant transverse principal symbol. By [14], the operator D is essentially self-adjoint with initial domain $C^\infty(M, E)$. Define an unbounded linear operator $\langle D\rangle$ in the space $L^2(M, E)$ as

$$\langle D\rangle = (D^2 + I)^{1/2}.$$

By the spectral theorem, the operator $\langle D\rangle$ is well-defined as a positive, self-adjoint operator in $L^2(M, E)$. The operator $\langle D\rangle^2 \in \Psi^2(M, E)$ is a bounded operator from $H^2(M, E)$ to $L^2(M, E)$. Hence, by interpolation, $\langle D\rangle$ defines a bounded operator from $H^1(M, E)$ to $L^2(M, E)$ and $H^1(M, E)$ is contained in the domain of $\langle D\rangle$ in $L^2(M, E)$.

By the spectral theorem, the operator $\langle D\rangle^s = (D^2 + I)^{s/2}$ is a well-defined positive self-adjoint operator in $\mathcal{H} = L^2(M, E)$ for any $s \in \mathbb{R}$, which is unbounded if $s > 0$. For any $s \geq 0$, denote by $\mathcal{H}^s$ the domain of $\langle D\rangle^s$, and, for $s < 0$, $\mathcal{H}^s = (\mathcal{H}^{-s})^*$. Put also $\mathcal{H}^\infty = \bigcap_{s\geq 0} \mathcal{H}^s$, $\quad \mathcal{H}^{-\infty} = (\mathcal{H}^\infty)^*$. It is clear that $H^s(M, E) \subset \mathcal{H}^s$ for any $s \geqslant 0$ and $\mathcal{H}^s \subset H^s(M, E)$ for any $s < 0$. In particular, $C^\infty(M, E) \subset \mathcal{H}^s$ for any s.

We say that a bounded operator A in $\mathcal{H}^\infty$ belongs to $\mathcal{L}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ (resp. $\mathcal{K}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$), if, for any s and r, it extends to a bounded (resp. compact) operator from $\mathcal{H}^s$ to $\mathcal{H}^r$, or, equivalently, the operator $\langle D\rangle^r A \langle D\rangle^{-s}$ extends to a bounded (resp. compact) operator in $L^2(M, E)$. It is easy to see that $\mathcal{L}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ is a involutive subalgebra in $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ is its ideal. We also introduce the class $\mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$, which consists of all operators from $\mathcal{K}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ such that, for any s and r, the operator $\langle D\rangle^r A \langle D\rangle^{-s}$ is a trace class operator in $L^2(M, E)$. It should be noted that any operator K with the smooth kernel belongs to $\mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$.

As an operator acting on half-densities, any operator $P \in \Psi^m(M)$ has the subprincipal symbol which is the well-defined homogeneous of degree $m - 1$ smooth function on $T^*M \setminus 0$ given in local coordinates by the formula

$$p_{sub} = p_{m-1} - \frac{1}{2i} \sum_{j=1}^{n} \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}, \tag{4}$$

where p_{m-1} and p_m are the homogeneous components of the complete symbol of P of degree $m - 1$ and m respectively. Observe that $p_{sub} = 0$ if P is a real, self-adjoint, differential operator of even order. In particular, this holds for the transversal Laplacian Δ_H on functions.

By the spectral theorem, the operator $\langle D\rangle$ defines a strongly continuous group $e^{it\langle D\rangle}$ of bounded operators in $L^2(M, E)$. Consider a one-parameter group Φ_t of $*$-automorphisms of the algebra $\mathcal{L}(L^2(M, E))$ defined by

$$\Phi_t(T) = e^{it\langle D\rangle} T e^{-it\langle D\rangle}, \quad T \in \mathcal{L}(L^2(M, E)).$$

The main result of the paper is the following theorem.

Theorem 1.7. *Let $D \in \Psi^1(M,E)$ be a formally self-adjoint, transversally elliptic operator such that D^2 has the scalar principal symbol and the holonomy invariant transverse principal symbol.*

(1) For any $K \in \Psi^{m,-\infty}(M,\mathcal{F},E)$, there exists an operator $K(t) \in \Psi^{m,-\infty}(M,\mathcal{F},E)$ such that $\Phi_t(K) - K(t), t \in \mathbb{R}$, is a smooth family of operators of class $\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^{\infty})$.

(2) If, in addition, E is the trivial line bundle, and the subprincipal symbol of D^2 is zero, then, for any $K \in \Psi^{m,-\infty}(M,\mathcal{F})$ with the principal symbol $k \in S^m(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$, the principal symbol $k(t) \in S^m(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ of the operator $K(t)$ is given by $k(t) = F_t^*(k)$, where F_t^* is the transverse bicharacteristic flow of the operator $\langle D\rangle$.

Remark 1.8. Theorem 1.7 implies Egorov's theorem for elliptic operators on compact Riemannian orbifolds. An m-dimensional orbifold M is a Hausdorff, second countable topological space, which is locally diffeomorphic to the quotient of $\mathbb{R}^m$ by a finite group of diffeomorphisms Γ. The notion of orbifold was first introduced by Satake in [24], where a different name, V-manifold, was used. We refer the reader to [24, 12, 3] for expositions of orbifold theory. It is well-known (see, for instance, [13]) that any orbifold M is diffeomorphic to the orbifold of G orbits of an action of a compact Lie group G on a compact manifold P where the action has finite isotropy groups (actually, one can take P to be the orthogonal frame bundle of M and $G = O(m)$). The orbits of this action are the leaves of a foliation $\mathcal{F}$ on P. We will use a natural isomorphism of the space $C^\infty(M)$ with the space $C^\infty(P)^G$ of G invariant functions on P. A pseudodifferential operator A in $C^\infty(P)$ can be defined as an operator acting on $C^\infty(P)^G$ which is the restriction of a G equivariant pseudodifferential operator $\tilde{A}$ in $C^\infty(M)$. The operator A is elliptic iff the corresponding operator $\tilde{A}$ is transversally elliptic with respect to the foliation $\mathcal{F}$. The orthogonal projection Π on the space of G-invariant functions in $C^\infty(P)$ is a transversal pseudodifferential operator of class $\Psi^{0,-\infty}(P,\mathcal{F})$. It follows that a pseudodifferential operator A in $C^\infty(M)$ coincides with the restriction of the operator $\Pi\tilde{A}\Pi \in \Psi^{0,-\infty}(P,\mathcal{F})$ to $C^\infty(P)^G$.

Fix Riemannian metrics g_M on M and g_P on P such that the quotient map $P \to M$ is a Riemannian submersion. So g_P is a bundle-like metric on the foliated manifold $(P,\mathcal{F})$. One can show that the associated transverse Laplacian Δ_H is G-invariant and the Laplacian Δ_M on M coincides with the restriction of Δ_H to $C^\infty(P)^G$. Therefore, we have

$$\begin{aligned} e^{it(\Delta_M+I)^{1/2}} A e^{-it(\Delta_M+I)^{1/2}} &= \Pi e^{it(\Delta_H+I)^{1/2}} \tilde{A} e^{-it(\Delta_H+I)^{1/2}} \Pi \\ &= \Pi e^{it(\Delta_H+I)^{1/2}} (\Pi\tilde{A}\Pi) e^{-it(\Delta_H+I)^{1/2}} \Pi. \end{aligned}$$

By Theorem 1.7, it follows that the operator $e^{it(\Delta_M+I)^{1/2}}Ae^{-it(\Delta_M+I)^{1/2}}$ is a pseudodifferential operator on M and one can describe its principal symbol as in the classical Egorov's theorem. The details will be given elsewhere.

1.4. **Noncommutative geodesic flow on foliated manifolds.** As stated in [14], any operator D, satisfying the assumptions of Section 1.3, defines a spectral triple in the sense of Connes' noncommutative geometry. In this setting, Theorem 1.7 has a natural interpretation in terms of the corresponding noncommutative geodesic flow. First, we recall general definitions [6, 5].

Let $(\mathcal{A},\mathcal{H},D)$ be a spectral triple [5]. Here

(1) $\mathcal{A}$ is an involutive algebra;
(2) $\mathcal{H}$ is a Hilbert space equipped with a $*$-representation of the algebra $\mathcal{A}$ (we will identify an element $a\in\mathcal{A}$ with the corresponding operator in $\mathcal{H}$);
(3) D is an (unbounded) self-adjoint operator in $\mathcal{H}$ such that
    (a): for any $a\in\mathcal{A}$, the operator $a(D-i)^{-1}$ is a compact operator in $\mathcal{H}$;
    (b): D almost commutes with any $a\in\mathcal{A}$ in the sense that $[D,a]$ is bounded in $\mathcal{H}$.

As above, let $\langle D\rangle = (D^2+I)^{1/2}$. By δ, we denote the (unbounded) derivative on $\mathcal{L}(\mathcal{H})$ given by

$$\delta(T)=[\langle D\rangle, T],\quad T\in \operatorname{Dom}\delta\subset\mathcal{L}(\mathcal{H}). \tag{5}$$

Let OP^α be the space of operators in $\mathcal{H}$ of order α, that means that $P\in\mathrm{OP}^\alpha$ iff $P\langle D\rangle^{-\alpha}\in\bigcap_n \operatorname{Dom}\,\delta^n$. In particular, $\mathrm{OP}^0=\bigcap_n \operatorname{Dom}\,\delta^n$. Denote by OP^0_0 the space of all operators $P\in\mathrm{OP}^0$ such that $\langle D\rangle^{-1}P$ and $P\langle D\rangle^{-1}$ are compact operators in $\mathcal{H}$. We also say that $P\in\mathrm{OP}^\alpha_0$ if $P\langle D\rangle^{-\alpha}$ and $\langle D\rangle^{-\alpha}P$ are in OP^0_0. It is easy to see that $\mathrm{OP}^{-\infty}_0=\bigcap_\alpha \mathrm{OP}^\alpha_0$ coincides with $\mathcal{K}(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$.

We will assume that $(\mathcal{A},\mathcal{H},D)$ is smooth. This means that, for any $a\in\mathcal{A}$, the bounded operators a and $[D,a]$ in $\mathcal{H}$ belong to OP^0. Let $\mathcal{B}$ be the algebra of bounded operators in $\mathcal{H}$ generated by the set of all operators of the form $\delta^n(a)$ with $a\in\mathcal{A}$ and $n\in\mathbb{N}$. Furthermore, we assume that the algebra $\mathcal{B}$ is contained in OP^0_0. In particular, this implies that $(\mathcal{B},\mathcal{H},D)$ is a spectral triple in the above sense.

In [6, 5], the definition of the algebra $\Psi^*(\mathcal{A})$ of pseudodifferential operators was given for a unital algebra $\mathcal{A}$. In the case under consideration, the algebra $\mathcal{A}$ is non-unital, that, roughly speaking, means that the associated geometric space is noncompact. Therefore, we must take into account behavior of pseudodifferential operators at "infinity". Next we define an algebra $\Psi^*_0(\mathcal{A})$, which can considered as an analogue of the algebra of pseudodifferential operators on a noncompact Riemannian manifold, whose symbols and all its derivatives of any order vanish at infinity. In particular, the assumptions on the spectral triple made above mean that the algebra $\mathcal{A}$ consists of smooth "functions", vanishing at "infinity" with all its derivatives of any order.

Define $\Psi_0^*(\mathcal{A})$ as the set of (unbounded) operators in $\mathcal{H}$, which admit an asymptotic expansion:

$$P \sim \sum_{j=0}^{+\infty} b_{q-j} \langle D\rangle^{q-j}, \quad b_{q-j} \in \mathcal{B}, \tag{6}$$

that means that, for any N,

$$P - \left(b_q\langle D\rangle^q + b_{q-1}\langle D\rangle^{q-1} + \ldots + b_{-N}\langle D\rangle^{-N}\right) \in \mathrm{OP}_0^{-N-1}.$$

By an easy modification of the proof of Theorem B.1 in [6, Appendix B], one can prove that $\Psi_0^*(\mathcal{A})$ is an algebra. Let $\mathcal{C}_0$ be the algebra $\mathcal{C}_0 = \mathrm{OP}_0^0 \bigcap \Psi_0^*(\mathcal{A})$, and $\bar{\mathcal{C}}_0$ the closure of $\mathcal{C}_0$ in $\mathcal{L}(\mathcal{H})$.

For any $T \in \mathcal{L}(\mathcal{H})$, define

$$\alpha_t(T) = e^{it\langle D\rangle} T e^{-it\langle D\rangle}, \quad t \in \mathbb{R}. \tag{7}$$

As usual, $\mathcal{K}$ denotes the ideal of compact operators in $\mathcal{H}$. The following definitions are motivated by the work of Connes [5].

Definition 1.9. Under the current assumptions on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, **the unitary cotangent bundle** $S^*\mathcal{A}$ is defined as the quotient of the C^*-algebra generated by all $\alpha_t(\bar{\mathcal{C}}_0), t \in \mathbb{R}$ and $\mathcal{K}$ by $\mathcal{K}$.

Definition 1.10. Under the current assumptions on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, **the noncommutative geodesic flow** is the one-parameter group α_t of automorphisms of the algebra $S^*\mathcal{A}$ defined by (7).

We consider spectral triples $(\mathcal{A}, \mathcal{H}, D)$ associated with a compact foliated Riemannian manifold $(M, \mathcal{F})$ [14]:

(1) The involutive algebra $\mathcal{A}$ is the algebra $C_c^\infty(G, |T\mathcal{G}|^{1/2})$;
(2) The Hilbert space $\mathcal{H}$ is the space $L^2(M, E)$ of L^2-sections of a holonomy equivariant Hermitian vector bundle E, on which an element k of the algebra $\mathcal{A}$ is represented via the $*$-representation R_E (see below for a definition);
(3) The operator D is a first order self-adjoint transversally elliptic operator with the holonomy invariant transversal principal symbol such that the operator D^2 has the scalar principal symbol.

We recall briefly the definitions of the structure of involutive algebra on $\mathcal{A}$ and of the representation R_E. Let $\alpha \in C^\infty(M, |T\mathcal{F}|^{1/2})$ be a strictly positive, smooth, leafwise half-density. One can lift α to a strictly positive, leafwise half-density $\nu^x = s^*\alpha \in C^\infty(G^x, |TG^x|^{1/2})$ via the covering map $s : G^x \to L_x$ (L_x is the leaf through a point $x \in M$). In the presence of ν, the space $\mathcal{A} = C_c^\infty(G, |T\mathcal{G}|^{1/2})$ is naturally identified with $C_c^\infty(G)$. We also assume, for simplicity, that there exists a holonomy invariant, smooth, transverse half-density $\Lambda \in C^\infty(M, |TM/T\mathcal{F}|^{1/2})$. Recall that the holonomy invariance of Λ means that $dh_\gamma^*(\Lambda(y)) = \Lambda(x)$ for any $\gamma \in G, s(\gamma) = x, r(\gamma) = y$, where the map $dh_\gamma^* : |T_yM/T_y\mathcal{F}|^{1/2} \to |T_xM/T_x\mathcal{F}|^{1/2}$ is induced by the corresponding linear holonomy map.

The multiplication and the involution in $\mathcal{A}$ are given by the formulas

$$\begin{aligned}(k_1 * k_2)(\gamma) &= \int_{G^x} k_1(\gamma'^{-1}\gamma)k_2(\gamma')\,d\nu^x(\gamma'), \quad \gamma\in G^x,\\ k^*(\gamma) &= \overline{k(\gamma^{-1})}, \quad \gamma\in G,\end{aligned} \tag{8}$$

where $k, k_1, k_2 \in \mathcal{A}$.

An Hermitian vector bundle E on M is holonomy equivariant, if it is equipped with an isometric action

$$T(\gamma) : E_x \to E_y, \quad \gamma\in G, \gamma : x\to y,$$

of G in fibers of E. Using the fixed half-densities α and Λ, one can identify elements of $L^2(M,E)$ with square integrable sections of the bundle E. Then, for any $u\in L^2(M,E)$, the section $R_E(k)u \in L^2(M,E)$ is defined by the formula

$$R_E(k)u(x) = \int_{G^x} k(\gamma)\,T(\gamma)[u(s(\gamma))]\,d\nu^x(\gamma), \quad x\in M.$$

It was stated in [14] that the spectral triple $(\mathcal{A},\mathcal{H},D)$ associated with a compact foliated Riemannian manifold is smooth. Recall that this means that, for any $a\in\mathcal{A}$, a and $[D,a]$ belong to $\mathrm{OP}^0=\bigcap_n \operatorname{Dom}\delta^n$. There is a gap in the proof of this fact given in [14]. In this paper, we give a correct proof (cf. Theorem 3.2 below). In Theorem 3.2, we also prove that, in the case in question, the algebra $\mathcal{B}$ mentioned above is contained in OP^0_0.

For any $\nu\in\tilde{N}^*\mathcal{F}$, there is a natural $*$-representation R_ν of the algebra $S^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ in $L^2(G^\nu_{\mathcal{F}_N}, s^*_N(\pi^*E))$. For its definition, we will use the strictly positive, leafwise half-density $\mu^\nu\in C^\infty(G^\nu_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ induced by α and the corresponding isomorphism $S^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})\cong S^0(G_{\mathcal{F}_N})$. Since E is a holonomy equivariant vector bundle, the bundle π^*E is also holonomy equivariant. The action of $G_{\mathcal{F}_N}$ in fibers of π^*E,

$$\pi^*T(\gamma,\nu) : (\pi^*E)_{dh^*_\gamma(\nu)} \to (\pi^*E)_\nu, \quad (\gamma,\nu)\in G_{\mathcal{F}_N},$$

is given by the formula $\pi^*T(\gamma,\nu)=T(\gamma)$, where we use the natural isomorphisms $(\pi^*E)_{dh^*_\gamma(\nu)}=E_x$ and $(\pi^*E)_\nu=E_y$. For any $k\in S^0(G_{\mathcal{F}_N})$ and $u\in L^2(G^\nu_{\mathcal{F}_N}, s^*_N(\pi^*E))$, the section $R_\nu(k)u\in L^2(G^\nu_{\mathcal{F}_N}, s^*_N(\pi^*E))$ is given by the formula

$$R_\nu(k)u(\gamma,\nu) = \int_{G^\nu_{\mathcal{F}_N}} k((\gamma',\nu)^{-1}(\gamma,\nu))\pi^*T(\gamma',\nu)[u(\gamma',\nu)]\,d\mu^\nu(\gamma',\nu),$$
$$(\gamma,\nu)\in G^\nu_{\mathcal{F}_N}.$$

It follows from the direct integral decomposition

$$L^2(G_{\mathcal{F}_N}, s^*_N(\pi^*E)) = \int_{N^*\mathcal{F}} L^2(G^\nu_{\mathcal{F}_N}, s^*_N(\pi^*E))\,d\nu,$$

that, for any $k\in S^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$, the continuous family

$$\{R_\nu(k)\in\mathcal{L}(L^2(G^\nu_{\mathcal{F}_N}, s^*_N(\pi^*E))) : \nu\in\tilde{N}^*\mathcal{F}\}$$

defines a bounded operator in $L^2(G_{\mathcal{F}_N}, s_N^*(\pi^* E))$. We will identify $k \in S^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ with the corresponding operator in $L^2(G_{\mathcal{F}_N}, s_N^*(\pi^* E))$ and denote by $\bar{S}^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ the closure of $S^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ in the uniform operator topology of $\mathcal{L}(L^2(G_{\mathcal{F}_N}, s_N^*(\pi^* E)))$. The transversal bicharacteristic flow F_t^* of the operator $\langle D\rangle$ extends by continuity to a strongly continuous one-parameter group of automorphisms of $\bar{S}^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$.

The following theorem gives a description of the associated noncommutative geodesic flow in the scalar case.

Theorem 1.11. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple associated with a compact foliated Riemannian manifold $(M, \mathcal{F})$ as above with E, being the trivial holonomy equivariant line bundle. Assume that the subprincipal symbol of D^2 vanishes. There exists a surjective homomorphism of involutive algebras $P : S^*\mathcal{A} \to \bar{S}^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ such that the following diagram commutes:*

$$\begin{array}{ccc} S^*\mathcal{A} & \xrightarrow{\alpha_t} & S^*\mathcal{A} \\ {\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle P} \\ \bar{S}^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2}) & \xrightarrow{F_t^*} & \bar{S}^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2}) \end{array} \tag{9}$$

2. Proof of the main theorem

2.1. **The case of elliptic operator.** Let $(M, \mathcal{F})$ be a compact foliated manifold, E a Hermitian vector bundle on M. In this section, we will assume that $D \in \Psi^1(M, E)$ is a formally self-adjoint, *elliptic* operator such that D^2 has the scalar principal symbol and the holonomy invariant transverse principal symbol. Then $P = \langle D\rangle \in \Psi^1(M, E)$ is a self-adjoint elliptic operator with the positive, scalar principal symbol p and the holonomy invariant transversal principal symbol. In this case, the elliptic extension $\tilde{p}$ of p introduced in Section 1.2 can be taken to be equal to p, $\tilde{p} = p$. Therefore, if we denote by X_p the Hamiltonian vector field of p on T^*M, then the vector field $\mathcal{H}_p$ can be described as a unique vector field on $G_{\mathcal{F}_N}$ such that $ds_N(\mathcal{H}_p) = X_p$ and $dr_N(\mathcal{H}_p) = X_p$. Similarly, one can define the transverse bicharacteristic flow F_t^* of P as in Definition 1.2, using p instead of $\tilde{p}$. The following theorem is slightly stronger than Theorem 1.7.

Theorem 2.1. *For any $K \in \Psi^{0,-\infty}(M, \mathcal{F}, E)$, the operator*

$$\Phi_t(K) = e^{itP} K e^{-itP}$$

is an operator of class $\Psi^{0,-\infty}(M, \mathcal{F}, E)$.

If E is the trivial line bundle, and the subprincipal symbol of D^2 vanishes, then, for any operator $K \in \Psi^{0,-\infty}(M, \mathcal{F})$ with the principal symbol $k \in S^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$, the operator $\Phi_t(K)$ has the principal symbol $k(t) \in S^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ given by $k(t) = F_t^*(k)$.

Proof. For the proof, we use theory of Fourier integral operators (see, for instance, [11, 26, 27]). Recall that a Fourier integral operator on M is a

linear operator $F : C^\infty(M) \to \mathcal{D}'(M)$, represented microlocally in the form

$$Fu(x) = \int e^{\phi(x,y,\theta)} a(x,y,\theta)\, u(y)\, dy\, d\theta, \tag{10}$$

where $x \in X \subset \mathbb{R}^n, y \in Y \subset \mathbb{R}^n, \theta \in \mathbb{R}^N \setminus 0$. Here $a(x,y,\theta) \in S^m(X \times Y \times \mathbb{R}^N)$ is an amplitude, ϕ is a non-degenerate phase function.

Consider the smooth map from $X \times Y \times \mathbb{R}^N$ to $T^*X \times T^*Y$ given by

$$(x,y,\theta) \mapsto (x, \phi_x(x,y,\theta), y, -\phi_y(x,y,\theta)).$$

The image of the set

$$\Sigma_\phi = \{(x,y,\theta) \in X \times Y \times \mathbb{R}^N : \phi_\theta(x,y,\theta) = 0\}$$

under this map turns out to be a homogeneous canonical relation Λ_ϕ in $T^*X \times T^*Y$. (Recall that a closed conic submanifold $C \in T^*(X \times Y) \setminus 0$ is called a homogeneous canonical relation, if it is Lagrangian with respect to the 2-form $\omega_X - \omega_Y$, where ω_X, ω_Y are the canonical symplectic forms in T^*X, T^*Y accordingly.)

The Fourier integral operator F given by the formula (10) is said to be associated with Λ_ϕ. We will write $F \in I^m(X \times Y, \Lambda_\phi)$, if $a \in S^{m+n/2-N/2}(X \times Y \times \mathbb{R}^N)$.

Operators from $\Psi^{m,-\infty}(M,\mathcal{F},E)$ can be described as Fourier integral operators associated with the immersed canonical relation $G'_{\mathcal{F}_N}$, which is the image of $G_{\mathcal{F}_N}$ under the mapping $G_{\mathcal{F}_N} \to T^*M \times T^*M : (\gamma,\nu) \mapsto (r_N(\gamma,\nu), -s_N(\gamma,\nu))$ [14]. Indeed, consider an elementary operator $A : C^\infty_c(U, E|_U) \to C^\infty_c(U', E|_{U'})$ given by the formula (1) with $k \in S^m(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$. It can be represented in the form (10), if we take $X = U$ with coordinates (x,y), $Y = U'$ with coordinates (x',y'), $\theta = \eta$, $N = q$, a phase function $\phi(x,y,x',y') = (y-y')\eta$ and an amplitude $a = k(x,x',y,\eta)$. The associated homogeneous canonical relation Λ_ϕ is the set of all $(x,y,\xi,\eta,x',y',\xi',\eta') \in T^*U \times T^*U'$ such that $y = y', \xi = \xi' = 0, \eta = -\eta'$, that coincides with the intersection of $G'_{\mathcal{F}_N}$ with $T^*U \times T^*U'$. Moreover, we see that

$$\Psi^{m,-\infty}(M,\mathcal{F},E) \subset I^{m-p/2}(M \times M, G'_{\mathcal{F}_N}; \mathcal{L}(E) \otimes |T(M \times M)|^{1/2}).$$

Since $G_{\mathcal{F}_N}$ is, in general, an immersed canonical relation, it is necessary to be more precise in the definition of the classes $I^m(M \times M, G'_{\mathcal{F}_N}; \mathcal{L}(E) \otimes |T(M \times M)|^{1/2})$. This can be done by analogy with the definition of the classes of longitudinal pseudodifferential operators on a foliated manifold given in [4] (see also [14] and the definition of classes $\Psi^{0,-\infty}(M,\mathcal{F},E)$ given above).

Let p be the principal symbol of P, and let $\Lambda_p(t), t \in \mathbb{R}$, be the canonical relation in $T^*M \times T^*M$ defined as

$$\Lambda_p(t) = \{((x,\xi),(y,\eta)) \in T^*M \times T^*M : (x,\xi) = f_{-t}(y,\eta)\},$$

where f_t is the Hamiltonian flow of p. It is well-known (cf., for instance, [26]) that e^{itP} is a Fourier integral operator associated with $\Lambda_p(t)$:

$$e^{itP} \in I^0(M \times M, \Lambda_p(t); \mathcal{L}(E) \otimes |T(M\times M)|^{1/2}).$$

By holonomy invariance of the transverse principal symbol of P, it follows that $\Lambda_p(t) \circ G_{\mathcal{F}_N} \circ \Lambda_p(-t) = G_{\mathcal{F}_N}$, and by the composition theorem of Fourier integral operators (see, for instance, [11]), we have $\Phi_t(K) = e^{itP}Ke^{-itP} \in \Psi^{0,-\infty}(M,\mathcal{F},E)$.

Now assume, in addition, that E is the trivial line bundle, the subprincipal symbol of D^2 vanishes, and $K \in \Psi^{0,-\infty}(M,\mathcal{F})$ with the principal symbol $k \in S^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$. Denote by $\mathcal{L}_{\mathcal{H}_p}$ the Lie derivative on $C^\infty(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ by the vector field $\mathcal{H}_p$. So the function $k(t) = F_t^*(k) \in S^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ is the solution of the equation

$$\frac{dk(t)}{dt} = \mathcal{L}_{\mathcal{H}_p} k(t), \quad t \in \mathbb{R},$$

with the initial data $k(0) = k$. By [10] (cf. also [7, 11]), it follows that, for any $K_1 \in \Psi^{0,-\infty}(M,\mathcal{F})$, the operator $[P, K_1]$ belongs to $\Psi^{0,-\infty}(M,\mathcal{F})$, and

$$\sigma([P,K_1]) = \frac{1}{i}\mathcal{L}_{\mathcal{H}_p}\sigma(K_1).$$

Consider any smooth family $\mathcal{K}(t) \in \Psi^{0,-\infty}(M,\mathcal{F}), t \in \mathbb{R}$, of operators with the principal symbol $k(t)$. Then

$$\begin{aligned}\frac{d\mathcal{K}(t)}{dt} &= i[P,\mathcal{K}(t)] + R(t), \quad t \in \mathbb{R},\\ \mathcal{K}(0) &= K + R_0,\end{aligned}$$

where $R(t) \in \Psi^{-1,-\infty}(M,\mathcal{F}), t \in \mathbb{R}$, is a smooth family of operators, and $R_0 \in \Psi^{-1,-\infty}(M,\mathcal{F})$.

Using the fact that $\Phi_t(K)$ is the solution of the Cauchy problem

$$\begin{aligned}\frac{d\Phi_t(K)}{dt} &= i[P,\Phi_t(K)], \quad t \in \mathbb{R},\\ \Phi_0(K) &= K,\end{aligned}$$

and the first part of the theorem, we get

$$\mathcal{K}(t) - \Phi_t(K) = \int_0^t \Phi_{t-\tau}(R(\tau))\,d\tau + \Phi_t(R_0) \in \Psi^{-1,-\infty}(M,\mathcal{F}),$$

and $\sigma(\Phi_t(K)) = \sigma(\mathcal{K}(t)) = k(t)$. □

2.2. **The general case.** In this section, we will prove Theorem 1.7 in the general case. Thus, we assume that $D \in \Psi^1(M,E)$ is a formally self-adjoint, transversally elliptic operator such that D^2 has the scalar principal symbol and the holonomy invariant transverse principal symbol.

Definition 2.2. An operator $A \in \Psi^l(M,E)$ is said to be of order $-\infty$ in some conic neighborhood of $N^*\mathcal{F}$, if, in any regular foliated chart with the coordinates $(x,y) \in I^p \times I^q$, there exists $\varepsilon > 0$ such that, for any multiindices α and β and for any natural N, its complete symbol $a \in S^l(I^n \times \mathbb{R}^n)$ satisfies the estimate with some constant $C_{\alpha\beta N} > 0$

$$|\partial_\xi^\alpha \partial_x^\beta a(x,y,\xi,\eta)| < C_{\alpha\beta N}(1+|\xi|+|\eta|)^{-N},$$
$$(x,y) \in I^p \times I^q, \quad (\xi,\eta) \in \mathbb{R}^p \times \mathbb{R}^q, \quad |\xi| < \varepsilon|\eta|.$$

The important fact, concerning to operators of order $-\infty$ in some conic neighborhood of $N^*\mathcal{F}$, is contained in the following lemma [14]:

Lemma 2.3. *If $A \in \Psi^l(M,E)$ is of order $-\infty$ in some conic neighborhood of $N^*\mathcal{F}$ and $K \in \Psi^{m,-\infty}(M,\mathcal{F},E)$, then AK and KA are in $\Psi^{-\infty}(M,E)$.*

Denote by $\mathcal{L}(\mathcal{D}'(M,E),\mathcal{H}^\infty)$ (resp. $\mathcal{L}(\mathcal{H}^{-\infty},C^\infty(M,E))$) the space of all bounded operators from $\mathcal{D}'(M,E)$ to $\mathcal{H}^\infty$ (resp. from $\mathcal{H}^{-\infty}$ to $C^\infty(M,E)$). Since any operator from $\Psi^{-N}(M,E)$ with $N > \dim M$ is a trace class operator in $L^2(M,E)$, one can easily show the following inclusions

$$\begin{aligned} \mathcal{L}(\mathcal{D}'(M,E),\mathcal{H}^\infty) &\subset \mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty), \\ \mathcal{L}(\mathcal{H}^{-\infty},C^\infty(M,E)) &\subset \mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty). \end{aligned} \tag{11}$$

Theorem 2.4. *For any $\alpha \in \mathbb{R}$, the operator $\langle D\rangle^\alpha = (D^2+I)^{\alpha/2}$ can be written as*

$$\langle D\rangle^\alpha = P(\alpha) + R(\alpha),$$

where:

(a) $P(\alpha) \in \Psi^\alpha(M,E)$ is a self-adjoint, elliptic operator with the positive, scalar principal symbol and the holonomy invariant transversal principal symbol;

(b) For any $K \in \Psi^{*,-\infty}(M,\mathcal{F},E)$, $KR(\alpha) \in \mathcal{L}(\mathcal{H}^{-\infty},C^\infty(M,E))$, and $R(\alpha)K \in \mathcal{L}(\mathcal{D}'(M,E),\mathcal{H}^\infty)$.

Proof. Using the standard construction of parametrix for elliptic operators in some conic neighborhood of $N^*\mathcal{F}$, one gets an analytic family $C_1(\lambda), \lambda \notin \mathbb{R}_+$, of operators from $\Psi^{-2}(M,E)$ such that

$$C_1(\lambda)(D^2+I-\lambda I) = I - r_1(\lambda), \quad \lambda \notin \mathbb{R}_+, \tag{12}$$

where $r_1(\lambda) \in \Psi^0(M,E)$ has order $-\infty$ in some conic neighborhood of $N^*\mathcal{F}$ (see [14] for more details). Hence, we have

$$(D^2+I-\lambda I)^{-1} = C_1(\lambda) + r_1(\lambda)(D^2+I-\lambda I)^{-1}, \quad \lambda \notin \mathbb{R}_+.$$

Using the Cauchy integral formula with an appropriate contour Γ in the complex plane, we get

$$(D^2+I)^{\alpha/2} = \frac{i}{2\pi}\int_\Gamma \lambda^{\alpha/2-N}\langle D\rangle^{2N}(D^2+I-\lambda I)^{-1}d\lambda = P_1(\alpha) + R_1(\alpha),$$

with some natural N such that $\operatorname{Re}\alpha < 2N$, where

$$P_1(\alpha) = \frac{i}{2\pi}\int_\Gamma \lambda^{\alpha/2-N}\langle D\rangle^{2N}C_1(\lambda)d\lambda,$$
$$R_1(\alpha) = \frac{i}{2\pi}\int_\Gamma \lambda^{\alpha/2-N}\langle D\rangle^{2N}r_1(\lambda)(D^2+I-\lambda I)^{-1}d\lambda.$$

In a standard manner (see [14]), one can prove that $P_1(\alpha)$ is a transversally elliptic operator of class $\Psi^\alpha(M,E)$ with the scalar principal symbol and the holonomy invariant, positive transversal principal symbol.

Let $K\in\Psi^{*,-\infty}(M,\mathcal{F},E)$. For any real s, one can write

$$KR_1(\alpha)\langle D\rangle^s = \frac{i}{2\pi}\int_\Gamma \lambda^{\alpha/2-N}K\langle D\rangle^{2N}r_1(\lambda)\langle D\rangle^s(D^2+I-\lambda I)^{-1}d\lambda.$$

By Lemma 2.3, the operator $K\langle D\rangle^{2N}r_1(\lambda)$ has the smooth kernel and defines a bounded operator from $\mathcal{H}^{-\infty}\subset\mathcal{D}'(M,E)$ to $C^\infty(M,E)$. Since $\langle D\rangle^s(D^2+I-\lambda I)^{-1}$ maps $\mathcal{H}^{-\infty}$ to $\mathcal{H}^{-\infty}$, this implies that the operator $KR_1(\alpha)$ is an operator of class $\mathcal{L}(\mathcal{H}^{-\infty},C^\infty(M,E))$.

Taking adjoints in (12), we get

$$(D^2+I-\lambda I)C_1^*(\lambda) = I - r_1^*(\lambda),\quad \lambda\notin\mathbb{R}_+.$$

It follows that $C_1(\lambda)-C_1^*(\lambda) = C_1(\lambda)r_1^*(\lambda)-r_1(\lambda)C_1^*(\lambda)$ has order $-\infty$ in some conic neighborhood of $N^*\mathcal{F}$. Moreover, using the formula

$$\begin{aligned}\langle D\rangle^{2N}C_1(\lambda)&-C_1^*(\lambda)\langle D\rangle^{2N}\\ &=\frac{1}{\lambda}\langle D\rangle^2\left(\langle D\rangle^{2(N-1)}C_1(\lambda)-C_1^*(\lambda)\langle D\rangle^{2(N-1)}\right)\langle D\rangle^2\\ &\quad+\frac{1}{\lambda}\left(\langle D\rangle^{2(N-1)}r_1(\lambda)-r_1^*(\lambda)\langle D\rangle^{2(N-1)}\right),\end{aligned}$$

one can prove by induction that $\langle D\rangle^{2N}C_1(\lambda)-C_1^*(\lambda)\langle D\rangle^{2N}$ has order $-\infty$ in some conic neighborhood of $N^*\mathcal{F}$. This implies that the same is true for $P_1(\alpha)-P_1^*(\alpha) = R_1^*(\alpha)-R_1(\alpha)$. Combining Lemma 2.3 and duality arguments, we get that, for any $K\in\Psi^{*,-\infty}(M,\mathcal{F},E)$, the operator $R_1(\alpha)K = (K^*R_1(\alpha)+K^*(R_1^*(\alpha)-R_1(\alpha)))^*$ extends to a bounded operator from $\mathcal{D}'(M,E)$ to $\mathcal{H}^\infty$.

Let $P(\alpha)\in\Psi^\alpha(M,E)$ be a self-adjoint, elliptic operator with the positive scalar principal symbol such that the operator $P_1(\alpha)-P(\alpha)$ has order $-\infty$ in some neighborhood of $N^*\mathcal{F}$ (see also [15]) and $R(\alpha)=\langle D\rangle^{\alpha/2}-P(\alpha)$. By Lemma 2.3, for any $K\in\Psi^{*,-\infty}(M,\mathcal{F},E)$, the operator $K(P(\alpha)-P_1(\alpha))$ is a smoothing operator, that immediately completes the proof. □

Let $\langle D\rangle = P+R$ be a representation given by Theorem 2.4. Denote by e^{itP} the strongly continuous group of bounded operators in $L^2(M,E)$ generated by the elliptic operator iP. Put also $R(t)=e^{it\langle D\rangle}-e^{itP}$.

Proposition 2.5. *For any $K \in \Psi^{*,-\infty}(M,\mathcal{F},E)$, $KR(t), t \in \mathbb{R}$, is a smooth family of operators from $\mathcal{L}(\mathcal{H}^{-\infty}, C^\infty(M,E))$, and $R(t)K, t \in \mathbb{R}$, is a smooth family of operators from $\mathcal{L}(\mathcal{D}'(M,E), \mathcal{H}^\infty)$.*

Proof. By the Duhamel formula, for any $K \in \Psi^{*,-\infty}(M,\mathcal{F},E)$ and $u \in H^1(M,E) \subset \mathrm{Dom}(P)$, one can write

$$KR(t)u = i\int_0^t e^{i\tau P} e^{-i\tau P} K e^{i\tau P}\, R\, e^{i(t-\tau)\langle D\rangle} u\, d\tau.$$

By Theorem 2.1, $e^{-i\tau P} K e^{i\tau P} \in \Psi^{*,-\infty}(M,\mathcal{F},E)$. Therefore, the operator $e^{-i\tau P} K e^{i\tau P} R$ extends to a bounded operator from $\mathcal{H}^{-\infty}$ to $C^\infty(M,E)$. Since $e^{i\tau P}$ maps $C^\infty(M,E)$ to $C^\infty(M,E)$ and $e^{i(t-\tau)\langle D\rangle}$ is a bounded operator in $\mathcal{H}^{-\infty}$, the operator $KR(t)$ extends to a bounded operator from $\mathcal{H}^{-\infty}$ to $C^\infty(M,E)$.

Using the formula

$$\frac{d^n}{dt^n} KR(t) = i\frac{d^{n-1}}{dt^{n-1}} KPR(t) + i^n KR\langle D\rangle^{n-1} e^{it\langle D\rangle}, \quad n \in \mathbb{N}, \tag{13}$$

one can show by induction that, for any $K \in \Psi^{*,-\infty}(M,\mathcal{F},E)$, the function $KR(t)$ is smooth as a function on $\mathbb{R}$ with values in $\mathcal{L}(\mathcal{H}^{-\infty}, C^\infty(M,E))$. The similar statement, concerning to the operator $R(t)K$, follows by duality. □

Proof of Theorem 1.7. Let $\langle D\rangle = P + R$ be a representation given by Theorem 2.4. Let $K \in \Psi^{m,-\infty}(M,\mathcal{F},E)$. By Theorem 2.1, it follows that the operator $\Phi_t^P(K) = e^{itP} K e^{-itP}$ is in $\Psi^{m,-\infty}(M,\mathcal{F},E)$. Moreover, if E is the trivial line bundle, the subprincipal symbol of D^2 vanishes, and $k \in S^m(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ is the principal symbol of K, then the principal symbol $k(t) \in S^m(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ of $\Phi_t^P(K)$ is given by $k(t) = F_t^*(k)$.

To complete the proof, it suffices to show that $\Phi_t(K) - \Phi_t^P(K), t \in \mathbb{R}$, is a smooth family of operators of class $\mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$. We have

$$\Phi_t(K) - \Phi_t^P(K) = e^{itP} KR(-t) + R(t) K e^{-it\langle D\rangle}.$$

Using Proposition 2.5, the fact that the operator e^{itP} takes $C^\infty(M,E)$ to itself and (11), we get that $e^{itP}KR(-t)$ belongs to $\mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$. Similarly, using Proposition 2.5, the fact that the operator $e^{-it\langle D\rangle}$ is bounded in $\mathcal{H}^{-\infty}$, and (11), we get that $R(t)K \in \mathcal{L}(\mathcal{D}'(M,E), \mathcal{H}^\infty) \subset \mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ and, furthermore, $R(t)Ke^{-it\langle D\rangle} \in \mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$. □

3. Noncommutative geometry of foliations

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple associated with a compact foliated Riemannian manifold $(M,\mathcal{F})$ as in Section 1.4. In this section, we give a description of all the objects introduced in Section 1.4 for this spectral triple. In particular, we will prove Theorem 1.11.

First, we introduce a notion of scalar principal symbol for an operator of class $\Psi^{m,-\infty}(M,\mathcal{F},E)$. Recall that the bundle π^*E on $N^*\mathcal{F}$ is holonomy

equivariant. Therefore, there is a canonical embedding

$$i : C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2}) \hookrightarrow C^\infty_{prop}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E)\otimes |T\mathcal{G}_N|^{1/2}),$$

which takes $k \in C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ to $i(k) = k\,\pi^*T$. We will identify $C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ with its image

$$i(C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})) \subset C^\infty_{prop}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E)\otimes |T\mathcal{G}_N|^{1/2}).$$

We say that $P \in \Psi^{m,-\infty}(M,\mathcal{F},E)$ has the scalar principal symbol if its principal symbol belongs to $C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$. Let $\Psi^{m,-\infty}_{sc}(M,\mathcal{F},E)$ denote the set of all $K \in \Psi^{m,-\infty}(M,\mathcal{F},E)$ with the scalar principal symbol. For any $k \in C^\infty_c(G, |T\mathcal{G}|^{1/2})$, the operator $R_E(k)$ is in $\Psi^{0,-\infty}_{sc}(M,\mathcal{F},E)$ and its principal symbol $\sigma(R_E(k))$ is equal to $\pi_G^* k \in C^\infty_{prop}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ where $\pi_G : G_{\mathcal{F}_N} \to G$ is defined in Section 1.1.

Recall that δ denotes the inner derivation on $\mathcal{L}(\mathcal{H})$ defined by $\langle D\rangle$ (see (5)). It is easy to see that the class $\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^{\infty})$ belongs to the domain of δ and is invariant under the action of δ. Moreover, one can easily show that $\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^{\infty})$ is an ideal in OP^0.

Proposition 3.1. *Any operator $K \in \Psi^{0,-\infty}(M,\mathcal{F},E)$ belongs to OP^0_0. Moreover, for any natural n and for any $K \in \Psi^{0,-\infty}(M,\mathcal{F},E)$, the operator $\delta^n(K)$ belongs to $\Psi^{0,-\infty}(M,\mathcal{F},E)+\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^{\infty})$. If $K \in \Psi^{0,-\infty}_{sc}(M,\mathcal{F},E)$, $\delta^n(K)$ belongs to $\Psi^{0,-\infty}_{sc}(M,\mathcal{F},E)+\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^{\infty})$.*

Proof. Let $\langle D\rangle = P + R$ be a representation given by Theorem 2.4. Let δ_0 denote the inner derivation on $\mathcal{L}(\mathcal{H})$ defined by P:

$$\delta_0(T) = [P,T], \quad T \in \mathrm{Dom}\,\delta_0 \subset \mathcal{L}(\mathcal{H}).$$

Let $K \in \Psi^{0,-\infty}(M,\mathcal{F},E)$. Since the principal symbol of P is scalar and its transversal principal symbol is holonomy invariant, it is easy to see that $\delta_0(K)$ is an operator of class $\Psi^{0,-\infty}(M,\mathcal{F},E)$, that implies that K belongs to the domain of δ_0^n for any natural n.

We will prove by induction on n that any $K \in \Psi^{0,-\infty}(M,\mathcal{F},E)$ belongs to the domain of δ^n for any natural n, and

$$\delta^n(K) - \delta_0^n(K) \in \mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^{\infty}).$$

By Theorem 2.4 and (11), it follows that

$$\delta(K) - \delta_0(K) = RK - KR \in \mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^{\infty}).$$

Now assume that the statement holds for some natural n. Then one can write

$$\delta^{n+1}(K) - \delta_0^{n+1}(K) = \delta(\delta^n(K) - \delta_0^n(K)) + R\delta_0^n(K) - \delta_0^n(K)R,$$

that belongs to $\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^{\infty})$, since δ takes $\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^{\infty})$ to itself and, by Theorem 2.4, $R\delta_0^n(K)$ and $\delta_0^n(K)R$ are in $\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^{\infty})$.

It remains to note that, by [14], for any $K \in \Psi^{0,-\infty}(M,\mathcal{F},E)$, the operators $K\langle D\rangle^{-1}$ and $\langle D\rangle^{-1}K$ are compact operators in $L^2(M,E)$. $\square$

Since $\mathcal{A} = C^\infty_c(G, |T\mathcal{G}|^{1/2}) \subset \Psi^{0,-\infty}(M,\mathcal{F},E)$, Proposition 3.1 easily implies the following

Theorem 3.2. *For any $a \in \mathcal{A}$, the operators a and $[D,a]$ belong to OP^0. Moreover, the algebra $\mathcal{B}$ generated by $\delta^n(a), a \in \mathcal{A}, n \in \mathbb{N}$ is contained in OP^0_0.*

By Theorem 3.2, it follows that the spectral triple $(\mathcal{A},\mathcal{H},D)$ is smooth. Next we will give a description of $\mathcal{B}$ and $\Psi^*_0(\mathcal{A})$.

Proposition 3.3. *Any element $b \in \mathcal{B}$ can written as $b = B + T$, where $B \in \Psi^{0,-\infty}_{sc}(M,\mathcal{F},E)$ and $T \in \mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$.*

Proof. By Proposition 3.1, the statement holds for any b of the form $\delta^n(a), a \in \mathcal{A}, n \in \mathbb{N}$. Since $\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$ is an ideal in OP^0, this implies the statement for an arbitrary element of $\mathcal{B}$. $\square$

Proposition 3.4. *For any natural N, the algebra $\Psi^*_0(\mathcal{A})$ is contained in $\Psi^{*,-\infty}_{sc}(M,\mathcal{F},E) + \mathrm{OP}^{-N}_0$.*

Proof. Take any $P \in \Psi^*_0(\mathcal{A})$ of the form $P \sim \sum_{j=0}^{+\infty} b_{q-j}\langle D\rangle^{q-j}$ with $b_{q-j} \in \mathcal{B}$. Fix an arbitrary integer j. Let $\langle D\rangle^j = P(j) + R(j)$ be a representation given by Theorem 2.4. By Proposition 3.3, one can write $b_j = B_j + T_j$, where $B_j \in \Psi^{0,-\infty}_{sc}(M,\mathcal{F},E)$ and $T_j \in \mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$. So we have

$$b_j\langle D\rangle^j = B_jP(j) + B_jR(j) + T_j\langle D\rangle^j.$$

Here $B_jP(j) \in \Psi^{j,-\infty}_{sc}(M,\mathcal{F},E)$ (see [14]), $B_jR(j) \in \mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$ by Theorem 2.4 and $T_j\langle D\rangle^j \in \mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$ by the definition of $\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$. Thus, $b_j\langle D\rangle^j \in \Psi^{j,-\infty}_{sc}(M,\mathcal{F},E) + \mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$, that completes the proof. $\square$

Now we need the following result on continuity of the principal symbol map given by (2). Let E be a vector bundle on a compact foliated manifold $(M,\mathcal{F})$. Denote by $\bar\Psi^{0,-\infty}(M,\mathcal{F},E)$ the closure of $\Psi^{0,-\infty}(M,\mathcal{F},E)$ in the uniform topology of $\mathcal{L}(L^2(M,E))$.

Proposition 3.5. (*1*) *The principal symbol map*

$$\sigma : \Psi^{0,-\infty}(M,\mathcal{F},E) \to S^0(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E)\otimes|T\mathcal{G}_N|^{1/2})$$

extends by continuity to a homomorphism

$$\bar\sigma : \bar\Psi^{0,-\infty}(M,\mathcal{F},E) \to \bar S^0(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E)\otimes|T\mathcal{G}_N|^{1/2}).$$

(*2*) *The ideal $I_\sigma = \mathrm{Ker}\,\bar\sigma$ contains the ideal $\mathcal{K}$ of compact operators in $L^2(M,E)$.*

Proposition 3.5 can be proven by an easy adaptation of the proof of analogous fact for pseudodifferential operators on compact manifolds (see, for instance, [22, 25]).

Proof of Theorem 1.11. By Proposition 3.4, it follows that the algebra $\mathcal{C}_0$ is contained in $\Psi^{0,-\infty}(M,\mathcal{F})+\mathrm{OP}^{-N}(\mathcal{H}^{-\infty},\mathcal{H}^{\infty})$ for any N and its closure, $\bar{\mathcal{C}}_0$, is contained in $\bar{\Psi}^{0,-\infty}(M,\mathcal{F})+\mathcal{K}$. By Proposition 3.5, the principal symbol map $\bar{\sigma}$ induces a map $P : S^*\mathcal{A} \to \bar{S}^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$. By Theorem 1.7, it follows that the diagram (9) is commutative that completes the proof. $\square$

Remark 3.6. Suppose E is a holonomy equivariant vector bundle. Let $C^*_E(G)$ be the closure of $R_E(C^\infty_c(G,|T\mathcal{G}|^{1/2}))$ in the uniform operator topology of $\mathcal{L}(L^2(M,E))$ and $C^*_r(G)$ the reduced foliation C^*-algebra (see, for instance, [9]). By [9], there is a natural surjective projection $\pi_E : C^*_E(G) \to C^*_r(G)$. The map $\pi_G : G_{\mathcal{F}_N} \to G$ defines a natural embedding $C^*_r(G) \subset \bar{S}^0(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$. Since $R_E(k) \in \Psi^{0,-\infty}(M,\mathcal{F},E)$ for any $C^\infty_c(G,|T\mathcal{G}|^{1/2})$, $C^*_E(G)$ is contained in $\bar{\Psi}^{0,-\infty}(M,\mathcal{F},E)$. Moreover, the restriction of $\bar{\sigma}$ to $C^*_E(G)$ coincides with π_E. So the principal symbol map $\bar{\sigma}$ provides an extension of π_E to $\bar{\Psi}^{0,-\infty}(M,\mathcal{F},E)$. In particular, if $I_\sigma = \operatorname{Ker}\bar{\sigma}$ coincides with $\mathcal{K}$, then π_E is injective, and the holonomy groupoid G is amenable (see, for instance, [1]).

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References

[1] Anantharaman-Delaroche, C. and Renault, J.: *Amenable groupoids.* Monographies de L'Enseignement Mathématique, 36. L'Enseignement Mathématique, Geneva, 2000.

[2] Block, J. and Getzler, E.: Quantization of foliations, In: *Proc. of the XXth Intern. Conf. on Diff. Geom. Methods in Theoretical Physics. June 3-7, 1991, New York City, USA. Vol.1*, World Sci. Publishing, River Edge, NJ, 1992, pp. 471–487.

[3] Chen, W. and Ruan, Y.: Orbifold Gromov-Witten Theory, In: *Orbifolds in mathematics and physics (Madison, WJ, 2001)*, Contemp. Math. 310, Amer. Math. Soc., Providence, 2002, pp. 25 – 85.

[4] Connes, A.: Sur la théorie non-commutative de l'intégration, In: *Algèbres d'opérateurs*, Lecture Notes in Math. 725. Springer, Berlin, 1979, pp. 19–143.

[5] Connes, A.: Geometry from the spectral point of view, *Lett. Math. Phys.* **34** (1995), 203–238.

[6] Connes, A. and Moscovici, H.: The local index formula in noncommutative geometry, *Geom. and Funct. Anal.* **5** (1995), 174–243.

[7] Duistermaat, J. and Hörmander, L.: Fourier intergal operators II, *Acta Math.* **128** (1972), 183–269.

[8] Egorov, Ju. V.: The canonical transformations of pseudodifferential operators, *Uspehi Mat. Nauk* **24**(1969), No. 5, 235–236.

[9] Fack, T. and Skandalis, G.: Sur les représentations et ideaux de la C^*-algèbre d'un feuilletage, *J.Operator Theory* **8** (1982), 95–129.

[10] Guillemin, V. and Sternberg, S.: Some problems in integral geometry and some related problems in microlocal analysis, *Amer. J. Math.* **101** (1979), 915–959.

[11] Hörmander, L.: *The analysis of linear partial differential operators IV.* Springer, Berlin, 1986.

[12] Kawasaki, T.: The signature theorem for V-manifolds, *Topology* **17** (1978), 75–83.

[13] Kawasaki, T.: The index of elliptic operators over V-manifolds, *Nagoya Math. J.* **84** (1981), 135 – 157.
[14] Kordyukov, Yu. A.: Noncommutative spectral geometry of Riemannian foliations, *Manuscripta Math.* **94** (1997), 45–73.
[15] Kordyukov, Yu. A.: The trace formula for transversally elliptic operators on Riemannian foliations (Russian), *Algebra i Analiz* **12** (2000), no. 3, 81–105; translation in *St. Petersburg Math. J.* **12**(2001), no. 3, 407–422.
[16] Libermann, P. and Marle C.-M.: *Symplectic Geometry and Analytical Mechanics.* Reidel, Dordrecht, 1987.
[17] Lichnerowicz, A.: Variétés symplectiques et dynamique attachée à une sous variété, *C. R. Acad. Sc. Paris* **280** (1975), 523–527.
[18] Lichnerowicz, A.: Les variétés de Poisson et leurs algèbres de Lie associées, *J. Diff. Geom.* **12** (1977), 253–300.
[19] Mardsen, J. E. and Weinstein, A.: Reduction os symplectic manifolds with symmetry, *Rep. Math. Phys.* **5**(1), 121–130.
[20] Molino, P.: *Riemannian foliations.* Birkhäuser, Boston, 1988.
[21] O'Neill: The fundamental equations of a submersion, *Mich. Math. J.* **13** (1996), 459–469.
[22] Palais, R.S.: *Seminar on the Atiyah - Singer index theorem.* Princeton Univ. Press, Princeton, 1965.
[23] Reinhart, B. L.: *Differential Geometry of Foliations.* Springer, Berlin, 1983.
[24] Satake, I.: The Gauss-Bonnet theorem for V-manifolds, *J. Math. Soc. Japan* **9** (1957), 464 – 492.
[25] Seeley, R. T.: Integro-differential operators on vector bundles, *Trans. Amer. Math. Soc.* **117** (1965), 167 – 204.
[26] Taylor, M.: *Pseudodifferential Operators.* Princeton Univ. Press, Princeton, 1981.
[27] Trèves, F.: *Introduction to pseudodifferential operators and Fourier integral operators. Volume 2: Fourier integral operators.* Plenum Press, New York and London, 1980.
[28] Xu, Ping: Noncommutative Poisson algebras. *Amer. J. Math.* **116** (1994), 101–125.

Institute of Mathematics, Russian Academy of Sciences, Ufa, Russia
E-mail address: `yuri@imat.rb.ru`